Continuous-Time Quantum Walks on Trees in Quantum Probability Theory

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Abstract

A quantum central limit theorem for a continuous-time quantum walk on a homogeneous tree is derived from quantum probability theory. As a consequence, a new type of limit theorems for another continuous-time walk introduced by the walk is presented. The limit density is similar to that given by a continuous-time quantum walk on the one-dimensional lattice.

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I. INTRODUCTION

Two types of quantum walks, discrete-time or continuous-time, were introduced as the quantum mechanical extension of the corresponding classical random walks and have been extensively studied over the last few years, see [1, 2] for recent reviews. In this paper we consider a continuous-time quantum walk on a homogeneous tree in quantum probability theory. The walk is defined by identifying the Hamiltonian of the system with a matrix related to the adjacency matrix of the tree. Concerning continuous-time quantum walks, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] for examples.

Let $\mathbb{T}_M^{(p)}$ denote a homogeneous tree of degree p with M-generation. After we fix a root $o \in \mathbb{T}_M^{(p)}$, a stratification (distance partition) is introduced by the natural distance function in the following way:

$$\mathbb{T}_M^{(p)} = \bigcup_{k=0}^{M} V_k^{(p)}, \quad V_k^{(p)} = \{x \in \mathbb{T}_M^{(p)} : \partial(o,x) = k\}.$$

Here $\partial(x,y)$ stands for the length of the shortest path connecting x and y. Then

$$|V_0^{(p)}| = 1,\ |V_1^{(p)}| = p,\ |V_2^{(p)}| = p(p-1), \ldots, |V_k^{(p)}| = p(p-1)^{k-1}, \ldots,$$

where $|A|$ is the number of elements in a set A. The total number of points in M-generation, $|\mathbb{T}_M^{(p)}|$, is $p(p-1)^M - (p-1)$.

Let $H_M^{(p)}$ be a $|\mathbb{T}_M^{(p)}| \times |\mathbb{T}_M^{(p)}|$ symmetric matrix given by the adjacency matrix of the tree $\mathbb{T}_M^{(p)}$. The matrix is treated as the Hamiltonian of the quantum system. The (i,j) component of $H_M^{(p)}$ denotes $H_M^{(p)}(i,j)$ for $i,j \in \{0,1,\ldots,|\mathbb{T}_M^{(p)}|-1\}$. In our case, the diagonal component of $H_M^{(p)}$ is always zero, i.e., $H_M^{(p)}(i,i) = 0$ for any i. On the other hand, the diagonal component of corresponding matrix $H_{M,MB}^{(p)}$ investigated in [15] for $p=3$ case is not zero. For example, $H_{1,MB}^{(3)}(0,0) = -3,\ H_{1,MB}^{(3)}(1,1) = H_{1,MB}^{(3)}(2,2) = H_{1,MB}^{(3)}(3,3) = -1$.

The evolution of continuous-time quantum walk on the tree of M-generation, $\mathbb{T}_M^{(p)}$, is governed by the following unitary matrix:

$$U_M^{(p)}(t) = e^{itH_M^{(p)}}.$$

The amplitude wave function at time t, $|\Psi_M^{(p)}(t)\rangle$, is defined by

$$|\Psi_M^{(p)}(t)\rangle = U_M^{(p)}(t)|\Psi_M^{(p)}(0)\rangle.$$

In this paper we take $|\Psi_M^{(p)}(0)\rangle = [1, 0, 0, \ldots, 0]^T$ as an initial state, where T denotes the transposed operator.

The $(n+1)$-th coordinate of $|\Psi_M^{(p)}(t)\rangle$ is denoted by $|\Psi_M^{(p)}(n,t)\rangle$ which is the amplitude wave function at site n at time t for $n = 0, 1, \ldots, p(p-1)^M - p$. The probability finding the walker is at site n at time t on $\mathbb{T}_M^{(p)}$ is given by

$$P_M^{(p)}(n,t) = \langle \Psi_M^{(p)}(n,t) | \Psi_M^{(p)}(n,t) \rangle.$$

Then we define the continuous-time quantum walk $X_M^{(p)}(t)$ at time t on $\mathbb{T}_M^{(p)}$ by

$$P(X_M^{(p)}(t) = n) = P_M^{(p)}(n,t).$$

In a similar way, let $X_{M,MB}^{(p)}(t)$ be a quantum walk given by $H_{M,MB}^{(p)}$. As we stated before, $H_{M,MB}^{(p)}(i,i)$ depends on i for any finite M. However in $M \to \infty$ limit, the (i,i) component of the matrix becomes $-p$ for any i. Remark that the probability distribution of the continuous-time walk does not depend on the value of the diagonal component of the scalar matrix. Therefore the definitions of the walks imply that both quantum walks coincide in $M \to \infty$ limit, i.e.,

$$\lim_{M\to\infty} P(X_M^{(p)}(t) = n) = \lim_{M\to\infty} P(X_{M,MB}^{(p)}(t) = n),$$

for any t and n.

This paper is organized as follows. In Sec. 2, we review the quantum probabilistic approach and give preliminaries and some examples for the walk $X_M^{(p)}(t)$. A quantum central limit theorem as $p \to \infty$ is derived from quantum probability theory in Sec. 3. Finally we present a limit theorem for another continuous-time walk $Y(t)$ introduced as a p-limit walk of $X_\infty^{(p)}(t)$.

II. QUANTUM PROBABILISTIC APPROACH

A. Finite M case

Let $\mu_M^{(p)}$ denote the spectral distribution of our adjacency matrix $H_M^{(p)}$. From the general theory of an interacting Fock space (see [14, 17, 18, 19, 20, 21], for examples), the orthogonal polynomials $\{Q_n^{(p)}\}$ and $\{Q_n^{(p,*)}\}$ associated with $\mu_M^{(p)}$ satisfy the following three-term

recurrence relations with a Szegö-Jacobi parameter $(\{\omega_n\}, \{\alpha_n\})$ respectively:

$$Q_0^{(p)}(x) = 1, \;\; Q_1^{(p)}(x) = x - \alpha_1,$$
$$xQ_n^{(p)}(x) = Q_{n+1}^{(p)}(x) + \alpha_{n+1}Q_n^{(p)}(x) + \omega_n Q_{n-1}^{(p)}(x) \;\; (n \geq 1),$$

and

$$Q_0^{(p,*)}(x) = 1, \;\; Q_1^{(p),*}(x) = x - \alpha_2,$$
$$xQ_n^{(p,*)}(x) = Q_{n+1}^{(p,*)}(x) + \alpha_{n+2}Q_n^{(p,*)}(x) + \omega_{n+1} Q_{n-1}^{(p,*)}(x) \;\; (n \geq 1).$$

In our tree case,

$$\omega_1 = p, \; \omega_2 = \omega_3 = \cdots = \omega_M = p-1, \; \omega_{M+1} = \omega_{M+2} = \cdots = 0, \;\; \alpha_1 = \alpha_2 = \cdots = 0.$$

Then the Stieltjes transform $G_{\mu_M^{(p)}}$ of $\mu_M^{(p)}$ is given by

$$G_{\mu_M^{(p)}}(x) = \frac{Q_{n-1}^{(p,*)}(x)}{Q_n^{(p)}(x)},$$

where $n = |\mathbb{T}_M^{(p)}| = p(p-1)^M - (p-1)$.

The following result was shown in [14]:

$$|\Psi_M^{(p)}(V_k^{(p)}, t)\rangle = \frac{1}{\sqrt{|V_k^{(p)}|}} \int_{\mathbb{R}} \exp(itx)\, Q_k^{(p)} \mu_M^{(p)}(x)\, dx,$$

for $k = 0, 1, 2, \ldots$. Remark that $|V_k^{(p)}| = \omega_1 \omega_2 \cdots \omega_k = p(p-1)^{k-1}$ $(1 \leq k \leq M)$ and $|V_0^{(p)}| = 1$. It is important to note that

$$|\Psi_M^{(p)}(n, t)\rangle = \frac{1}{|V_k^{(p)}|} \int_{\mathbb{R}} \exp(itx)\, Q_k^{(p)}(x) \mu_M^{(p)}(x)\, dx \quad \text{if } n \in V_k^{(p)} \;\; (k = 0, 1, \ldots, M).$$

The proof appeared in Appendix A in [14].

B. $p = 3$ and $M = 2$ case

Here we consider $p = 3$ and $M = 2$ case. Then we have $n = 10$, $\omega_1 = 3, \omega_2 = 2, \omega_3 = \omega_4 = \cdots = 0$, $\alpha_1 = \alpha_2 = \cdots = 0$. The definitions of $Q_n^{(3)}(x)$ and $Q_n^{(3,*)}(x)$ imply

$$Q_0^{(3)}(x) = 1, \;\; Q_1^{(3)}(x) = x, \;\; Q_2^{(3)}(x) = x^2 - 3, \;\; Q_k^{(3)}(x) = x^{k-2}(x^2 - 5) \;\; (k \geq 3),$$

and

$$Q_0^{(3,*)}(x) = 1,\ \ Q_1^{(3,*)}(x) = x,\ \ Q_k^{(3,*)}(x) = x^{k-2}(x^2-2)\ \ (k \geq 2).$$

Therefore we obtain the Stieltjes transform:

$$G_{\mu_2^{(3)}}(x) = \frac{Q_9^{(3,*)}(x)}{Q_{10}^{(3)}(x)} = \frac{2}{5}\cdot\frac{1}{x} + \frac{3}{10}\cdot\frac{1}{x-\sqrt{5}} + \frac{3}{10}\cdot\frac{1}{x+\sqrt{5}}.$$

From this, we see that

$$\mu_2^{(3)} = \frac{2}{5}\,\delta_0(x) + \frac{3}{10}\,\delta_{-\sqrt{5}}(x) + \frac{3}{10}\,\delta_{\sqrt{5}}(x).$$

Then

$$\begin{aligned}
|\Psi_2^{(3)}(V_0^{(3)},t)\rangle &= \int_{\mathbb{R}} \exp(itx)\,\mu_2^{(3)}(dx) = \frac{1}{5}\,(2+3\cos(\sqrt{5}t)),\\
|\Psi_2^{(3)}(V_1^{(3)},t)\rangle &= \frac{1}{\sqrt{\omega_1}}\int_{\mathbb{R}} \exp(itx)Q_1^{(3)}(x)\,\mu_2^{(3)}(dx) = \frac{i\sqrt{3}}{\sqrt{5}}\,\sin(\sqrt{5}t),\\
|\Psi_2^{(3)}(V_2^{(3)},t)\rangle &= \frac{1}{\sqrt{\omega_1\omega_2}}\int_{\mathbb{R}} \exp(itx)Q_2^{(3)}(x)\,\mu_2^{(3)}(dx) = \frac{\sqrt{6}}{5}\left(-1+\cos(\sqrt{5}t)\right).
\end{aligned}$$

Noting that $|\Psi_2^{(3)}(n,t)\rangle = |\Psi_2^{(3)}(V_k^{(3)},t)\rangle/\sqrt{|V_k^{(3)}|}$ for any $k = 0,1,2$, we obtain the same conclusion as the result given by the eigenvalues and the eigenvectors of $H_2^{(3)}$.

C. $M \to \infty$ case

The quantum probabilistic approach [14, 20, 21] implies that

$$|\Psi_\infty^{(p)}(V_k^{(p)},t)\rangle = \lim_{M\to\infty}|\Psi_M^{(p)}(V_k,t)\rangle = \frac{1}{\sqrt{|V_k^{(p)}|}}\int_{\mathbb{R}} \exp(itx)\,Q_k^{(p)}(x)\mu_\infty^{(p)}(x)\,dx,$$

for $k = 0,1,2,\ldots$, where the limit spectral distribution $\mu_\infty^{(p)}(x)$ is given by

$$I_{(-2\sqrt{p-1},2\sqrt{p-1})}(x)\,\frac{p\sqrt{4(p-1)-x^2}}{2\pi(p^2-x^2)}.$$

Here I_A is the indicator function of A, i.e., $I_A(x) = 1$, if $x \in A$, $= 0$, if $x \notin A$. This type of measure was first obtained by Kesten [22] in a classical random walk with a different method. An immediate consequence is

$$P_\infty^{(p)}(V_k^{(p)},t) = \frac{1}{|V_k^{(p)}|}\left[\left\{\int_{\mathbb{R}}\cos(tx)\,Q_k^{(p)}(x)\mu_\infty^{(p)}(x)\,dx\right\}^2 + \left\{\int_{\mathbb{R}}\sin(tx)\,Q_k^{(p)}(x)\mu_\infty^{(p)}(x)\,dx\right\}^2\right],$$

for $k = 0, 1, 2, \ldots$ Furthermore, as in the case of finite M, we see that

$$|\Psi_{\infty}^{(p)}(n,t)\rangle = \frac{1}{|V_k^{(p)}|} \int_{\mathbb{R}} \exp(itx)\, Q_k^{(p)}(x) \mu_{\infty}^{(p)}(x)\, dx, \tag{1}$$

if $n \in V_k^{(p)}$ $(k = 0, 1, 2, \ldots)$. From (1) and the Riemann-Lebesgue lemma, we have $\lim_{t\to\infty} |\Psi_{\infty}^{(p)}(n,t)\rangle = 0$, for any n, since $Q_k^{(p)}(x)\mu_{\infty}^{(p)}(x) \in L^1(\mathbb{R})$. Therefore we see that $\lim_{t\to\infty} P_{\infty}^{(p)}(n,t) = 0$. So we conclude that $\bar{P}_{\infty}^{(p)}(n) = 0$, where $\bar{P}_{\infty}^{(p)}(n)$ is the time-averaged distribution of $P_{\infty}^{(p)}(n,t)$.

D. $p = 2$ and $M \to \infty$ case

In this subsection, we consider $p = 2$ and $M \to \infty$, i.e., $\mathbb{Z}^1$ case. Then we have

Proposition 1.

$$|\Psi_{\infty}^{(2)}(V_0^{(2)},t)\rangle = J_0(2t), \qquad |\Psi_{\infty}^{(2)}(V_k^{(2)},t)\rangle = \sqrt{2}\, i^k\, J_k(2t), \quad (k = 0, 1, 2, \ldots),$$

where $J_n(x)$ is the Bessel function of the first kind of order n.

Proof. We induct on k. For $k = 0$ case, we use the following result (see (4) in page 48 in [23]):

$$\int_{-1}^{1} \exp(isx)\ (1 - x^2)^{\nu - 1/2}\, dx = \frac{\Gamma(1/2)\Gamma(\nu + 1/2)}{(s/2)^{\nu}}\, J_{\nu}(s), \tag{2}$$

where $\Gamma(x)$ is the Gamma function. Combining $\Gamma(3/2) = \sqrt{\pi}/2, \Gamma(1/2) = \sqrt{\pi}$ with $Q_0^{(2)}(x) = 1$ and $\nu = 0$ gives

$$|\Psi_{\infty}^{(2)}(V_0^{(2)},t)\rangle = \int_{-2}^{2} \exp(itx)\ \frac{1}{\pi\sqrt{4 - x^2}}\, dx \ = \ J_0(2t).$$

In a similar fashion, we verify that the result holds for $k = 1, 2$.

Next we suppose that the result is true for all values up to k, where $k \geq 2$. Then we see

that

$$
\begin{aligned}
& |\Psi^{(2)}_\infty(V^{(2)}_{k+1},t)\rangle \\
&= \frac{1}{\sqrt{2}} \int_{-2}^{2} \exp(itx)\ Q^{(2)}_{k+1}(x)\ \frac{dx}{\pi\sqrt{4-x^2}} \\
&= \frac{1}{\sqrt{2}} \int_{-2}^{2} \exp(itx)\ \left\{ xQ^{(2)}_{k}(x) - Q^{(2)}_{k-1}(x) \right\}\ \frac{dx}{\pi\sqrt{4-x^2}} \\
&= \frac{1}{i}\frac{d}{dt}\left(\frac{1}{\sqrt{2}} \int_{-2}^{2} \exp(itx)\ Q^{(2)}_{k}(x)\ \frac{dx}{\pi\sqrt{4-x^2}} \right) - \frac{1}{\sqrt{2}} \int_{-2}^{2} \exp(itx)\ Q^{(2)}_{k-1}(x)\ \frac{dx}{\pi\sqrt{4-x^2}} \\
&= \frac{1}{i}\frac{d}{dt}(\sqrt{2}\, i^k\ J_k(2t)) - \sqrt{2}\, i^{k-1}\ J_{k-1}(2t) \\
&= \sqrt{2}\, i^{k+1}\ J_{k+1}(2t).
\end{aligned}
$$

The second equality follows from the definition of $Q^{(2)}_k(x)$. By induction, we have the fourth equality. For the last equality, we use a recurrence formula for the Bessel coefficients: $2J'_k(2t) = J_{k-1}(2t) - J_{k+1}(2t)$ (see (2) in page 17 of [23]). □

As a consequence, we have

Corollary 1.

$$
P^{(2)}_\infty(V^{(2)}_0,t) = J^2_0(2t), \qquad P^{(2)}_\infty(V^{(2)}_k,t) = 2J^2_k(2t), \quad (k=1,2,\ldots).
$$

We confirm that

$$
\sum_{k=0}^{\infty} P^{(2)}_\infty(V^{(2)}_k,t) = 1,
$$

since it follows from $J^2_0(2t) + 2\sum_{k=1}^{\infty} J^2_k(2t) = 1$ (see (3) in page 31 in [23]). Noting that $V^{(2)}_k = \{-k,k\}$ for any $k \geq 0$, we have the same result given by [10]:

$$
P^{(2)}_\infty(n,t) = J^2_n(2t),
$$

for any $n \in \mathbb{Z}$ and $t \geq 0$.

III. QUANTUM CENTRAL LIMIT THEOREM

To state a quantum central limit theorem in our case, it is convenient to rewrite as

$$
\left\langle \Phi^{(p)}_k \left| \exp\left(itH^{(p)}_\infty\right) \right| \Phi^{(p)}_0 \right\rangle = |\Psi^{(p)}_\infty(V^{(p)}_k,t)\rangle,
$$

where

$$\Phi_k^{(p)} = \frac{1}{\sqrt{|V_k^{(p)}|}} \sum_{n \in V_k^{(p)}} I_n,$$

and I_n denotes the indicator function of the singleton $\{n\}$. It is easily obtained that

$$\lim_{p\to\infty} \left\langle \Phi_k^{(p)} \left| \exp\left(itH_\infty^{(p)}\right) \right| \Phi_0^{(p)} \right\rangle = 0,$$

for any $k \geq 0$. Then we have the following quantum central limit theorem:

Theorem 1.

$$\lim_{p\to\infty} \left\langle \Phi_k^{(p)} \left| \exp\left(it\frac{H_\infty^{(p)}}{\sqrt{p}}\right) \right| \Phi_0^{(p)} \right\rangle = (k+1)\, i^k\, \frac{J_{k+1}(2t)}{t},$$

for $k = 0, 1, 2, \ldots$.

Proof. We induct on k. First we consider $k = 0$ case. We see that

$$\begin{aligned}
&\lim_{p\to\infty} \left\langle \Phi_0^{(p)} \left| \exp\left(it\frac{H_\infty^{(p)}}{\sqrt{p}}\right) \right| \Phi_0^{(p)} \right\rangle \\
&= \lim_{p\to\infty} \int_{\mathbb{R}} \exp\left(it\frac{x}{\sqrt{p}}\right)\, \mu_\infty^{(p)}(x)\, dx \\
&= \lim_{p\to\infty} \int_{-2\sqrt{(p-1)/p}}^{2\sqrt{(p-1)/p}} \exp\left(itx\right)\, \frac{\sqrt{(2(p-1)/p)^2 - x^2}}{2\pi(1-x^2/p)}\, dx \\
&= \int_{-1}^{1} \exp\left(2itx\right)\, \frac{2\sqrt{1-x^2}}{\pi}\, dx
\end{aligned}$$

Then (2) with $\nu = 1$ yields

$$\lim_{p\to\infty} \left\langle \Phi_0^{(p)} \left| \exp\left(it\frac{H_\infty^{(p)}}{\sqrt{p}}\right) \right| \Phi_0^{(p)} \right\rangle = \frac{J_1(2t)}{t}.$$

So the result holds for $k = 0$. Similarly we obtain

$$\begin{aligned}
\lim_{p\to\infty} \left\langle \Phi_1^{(p)} \left| \exp\left(it\frac{H_\infty^{(p)}}{\sqrt{p}}\right) \right| \Phi_0^{(p)} \right\rangle &= \frac{2iJ_2(2t)}{t}, \\
\lim_{p\to\infty} \left\langle \Phi_2^{(p)} \left| \exp\left(it\frac{H_\infty^{(p)}}{\sqrt{p}}\right) \right| \Phi_0^{(p)} \right\rangle &= -\frac{3J_3(2t)}{t}.
\end{aligned}$$

Next we suppose that the result holds for all values up to k, where $k \geq 2$. Then we have

$$\begin{aligned}
&\lim_{p\to\infty} \left\langle \Phi_{k+1}^{(p)} \left| \exp\left(it \frac{H_\infty^{(p)}}{\sqrt{p}} \right) \right| \Phi_0^{(p)} \right\rangle \\
&= \lim_{p\to\infty} \frac{1}{\sqrt{|V_{k+1}^{(p)}|}} \int_{\mathbb{R}} \exp\left(it \frac{x}{\sqrt{p}} \right) Q_{k+1}^{(p)}(x)\, \mu_\infty^{(p)}(x)\, dx \\
&= \lim_{p\to\infty} \frac{1}{\sqrt{p(p-1)^k}} \int_{-2\sqrt{(p-1)/p}}^{2\sqrt{(p-1)/p}} \exp(itx)\; Q_{k+1}^{(p)}(\sqrt{p}x)\, \frac{\sqrt{(2(p-1)/p)^2 - x^2}}{2\pi(1 - x^2/p)}\, dx \\
&= \int_{-2}^{2} \exp(itx)\; Q_{k+1}^{(\infty)}(x)\, \frac{\sqrt{2^2 - x^2}}{2\pi}\, dx \\
&= \int_{-2}^{2} \exp(itx)\; \left\{ xQ_k^{(\infty)}(x) - Q_{k-1}^{(\infty)}(x) \right\} \frac{\sqrt{2^2 - x^2}}{2\pi}\, dx \\
&= \frac{1}{i}\frac{d}{dt} \left(\int_{-1}^{1} \exp(2itx)\; Q_k^{(\infty)}(2x)\, \frac{2\sqrt{1-x^2}}{\pi}\, dx \right) \\
&\qquad\qquad - \int_{-1}^{1} \exp(2itx)\; Q_{k-1}^{(\infty)}(2x)\, \frac{2\sqrt{1-x^2}}{\pi}\, dx \\
&= i^{k-1} \left\{ (k+1)\frac{d}{dt}\left(\frac{J_{k+1}(2t)}{t} \right) - k\frac{J_k(2t)}{t} \right\}
\end{aligned}$$

where the last equality is given by the induction and $Q_k^{(\infty)}(x) = \lim_{p\to\infty} Q_k^{(p)}(\sqrt{p}x)/\sqrt{p(p-1)^{k-1}}$, if the right hand side exists. We confirm that the limit exists for any $k \geq 1$. For example, we compute $Q_1^{(\infty)}(x) = x$, $Q_2^{(\infty)}(x) = x^2 - 1$, $Q_3^{(\infty)}(x) = x^3 - 2x$, $Q_4^{(\infty)}(x) = x^4 - 3x^2 + 1, \ldots$ In order to prove the result, it suffices to check the following relation:

$$(k+1)\frac{d}{dt}\left(\frac{J_{k+1}(2t)}{t} \right) - k\frac{J_k(2t)}{t} = -(k+2)\frac{J_{k+2}(2t)}{t}.$$

The left hand side of this equation becomes

$$\begin{aligned}
&(k+1)\frac{2J_{k+1}'(2t)}{t} - (k+1)\frac{J_{k+1}(2t)}{t^2} - k\frac{J_k(2t)}{t} \\
&= \frac{J_k(2t)}{t} - (k+1)\frac{J_{k+1}(2t)}{t^2} - (k+1)\frac{J_k(2t)}{t} \\
&= -(k+2)\frac{J_{k+2}(2t)}{t},
\end{aligned}$$

since the first and second equalities are obtained from recurrence formulas for the Bessel coefficients: $2J_{k+1}'(2t) = J_k(2t) - J_{k+2}(2t)$ and $J_k(2t) + J_{k+2}(2t) = (k+1)J_{k+1}(2t)/t$ (see (1) in page 17 of [23]), respectively. This finishes the proof of the theorem. $\square$

IV. A NEW TYPE OF LIMIT THEOREMS

We can now state the main result of this paper. To do so, let define

$$|\tilde{\Psi}_{\infty}^{(\infty)}(V_k^{(\infty)},t)\rangle = \lim_{p\to\infty} \left\langle \Phi_k^{(p)} \left| \exp\left(it\frac{H_{\infty}^{(p)}}{\sqrt{p}} \right) \right| \Phi_0^{(p)} \right\rangle,$$

and

$$\tilde{P}_{\infty}^{(\infty)}(V_k^{(\infty)},t) \; = \; \langle \tilde{\Psi}_{\infty}^{(\infty)}(V_k^{(\infty)},t) | \tilde{\Psi}_{\infty}^{(\infty)}(V_k^{(\infty)},t) \rangle.$$

By Theorem 1 and the definition of $|\tilde{\Psi}_{\infty}^{(\infty)}(V_k^{(\infty)},t)\rangle$, we see that

$$\sum_{k=0}^{\infty} \tilde{P}_{\infty}^{(\infty)}(V_k^{(\infty)},t) = \sum_{k=1}^{\infty} k^2 \, \frac{J_k^2(2t)}{t^2} = 1. \tag{3}$$

The second equality comes from an expansion of z^2 as a series of squares of Bessel coefficients (see page 37 in [23]):

$$z^2 = 4 \sum_{k=1}^{\infty} k^2 \, J_k^2(z).$$

Noting the result (3), here we define another continuous-time quantum walk $Y(t)$ starting from the root defined by

$$P(Y(t)=k) = \tilde{P}_{\infty}^{(\infty)}(V_k^{(\infty)},t) = (k+1)^2 \, \frac{J_{k+1}^2(2t)}{t^2}.$$

Therefore we obtain

Theorem 2.

$$\frac{Y(t)}{t} \;\; \Rightarrow \;\; Z,$$

as $t \to \infty$, where $\Rightarrow$ means the weak convergence and Z has the following density function:

$$I_{(0,2)}(x) \, \frac{x^2}{\pi\sqrt{4-x^2}}.$$

Proof. From Theorem 1, we begin with computing

$$E\left(\exp\left(i\xi \frac{Y(t)}{t} \right) \right) \;\; = \;\; \frac{\exp(-i\xi/t)}{t^2} \sum_{k=1}^{\infty} \exp\left(i\xi \frac{k}{t} \right) k^2 J_{k+1}^2(2t),$$

for $\xi \in \mathbb{R}$. By Neumann's addition theorem (see p.358 in [23]), we have

$$J_0(\sqrt{a^2+b^2-2ab\cos(\xi)}) = \sum_{k=-\infty}^{\infty} J_k(a)J_k(b)\exp(ik\xi).$$

Taking $t=a=b$ in this equation gives

$$J_0(4t\sqrt{\sin(\xi/2)}) = \sum_{k=-\infty}^{\infty} J_k^2(t)\exp(ik\xi).$$

By differentiating both sides of the equation with respect to t twice, we see

$$\begin{aligned}\sum_{k=1}^{\infty} k^2 J_k^2(t)\exp(ik\xi) &= \frac{1}{2}\sum_{k=-\infty}^{\infty} k^2 J_k^2(t)\exp(ik\xi)\\ &= \frac{t}{4}\sin\left(\frac{\xi}{2}\right) J_0'\left(2t\sin\left(\frac{\xi}{2}\right)\right) - \frac{t^2}{2}\cos^2\left(\frac{\xi}{2}\right) J_0''\left(2t\sin\left(\frac{\xi}{2}\right)\right).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}E\left(\exp\left(i\xi\frac{Y(t)}{t}\right)\right) &= \exp\left(-\frac{i\xi}{t}\right)\left\{\frac{1}{2t}\sin\left(\frac{\xi}{2t}\right) J_0'\left(4t\sin\left(\frac{\xi}{2t}\right)\right)\right.\\ &\qquad\left. -2\cos^2\left(\frac{\xi}{2t}\right) J_0''\left(4t\sin\left(\frac{\xi}{2t}\right)\right)\right\}.\end{aligned}$$

Then a similar argument in [10] yields

$$\lim_{t\to\infty} E\left(\exp\left(i\xi\frac{Y(t)}{t}\right)\right) = -2J_0''(2\xi).$$

On the other hand, (2) with $\nu=0$ gives

$$J_0''(2\xi) = -\int_{-1}^{1}\exp(2i\xi x)\,\frac{x^2}{\pi\sqrt{1-x^2}}\,dx.$$

From the last two equations, we conclude that

$$\lim_{t\to\infty} E\left(\exp\left(i\xi\frac{Y(t)}{t}\right)\right) = \int_{0}^{2}\exp(i\xi x)\,\frac{x^2}{\pi\sqrt{4-x^2}}\,dx.$$

$\square$

It is interesting to remark that when $p=2$ case, i.e., $\mathbb{Z}^1$, a similar type of density function was derived from a limit theorem for $X(t)$ (see [10]):

$$\lim_{t\to\infty} E\left(\exp\left(i\xi\frac{X(t)}{t}\right)\right) = \int_{0}^{1}\exp(i\xi x)\,\frac{2}{\pi\sqrt{1-x^2}}\,dx.$$

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